AVERAGE CODE LENGTH

How to achieve progressively lower code lengths?

Recall, optimal codeword length for i^{th} symbol S_i

$$l_i = \log_2\left(\frac{1}{P_i}\right)$$

Need not be an integer

Choose $l_i = \lceil \log_2\left(\frac{1}{P_i}\right) \rceil$

integer ceiling Function

Gives the next highest integer

Ex: $\lceil 4.7 \rceil = 5$

$$\log_2\left(\frac{1}{P_i}\right) \leq \underbrace{\left\lceil \log_2\left(\frac{1}{P_i}\right) \right\rceil}_{l_i} \leq \log_2\left(\frac{1}{P_i}\right) + 1$$

Ex: $4.7 \leq \lceil 4.7 \rceil = 5 \leq 4.7 + 1 = 5.7$

$x \leq \lceil x \rceil \leq x + 1$

$$\underbrace{\sum_{i=1}^{M-1} P_i \log_2\left(\frac{1}{P_i}\right)}_{H(X)} \leq \underbrace{\sum_{i=0}^{M-1} P_i \underbrace{\left\lceil \log_2\left(\frac{1}{P_i}\right) \right\rceil}_{l_i}}_{L}$$

$$\leq \sum_{i=0}^{M-1} P_i \left(\log_2 \left(\frac{1}{P_i} \right) + 1 \right)$$

$$= \underbrace{\sum_{i=0}^{M-1} P_i \log_2 \left(\frac{1}{P_i} \right)}_{H(X)} + \underbrace{\sum_{i=0}^{M-1} P_i}_{1}$$

$$= H(X) + 1$$

We obtain,

$$\boxed{H(X) \leq L \leq H(X) + 1}$$

is within 1 bit of entropy

How to achieve a code length arbitrarily close to the Entropy?

Consider the sequence,

$$\bar{X} = \underbrace{X_1, X_2, \ldots, X_n}_{n \text{ symbols together}}$$

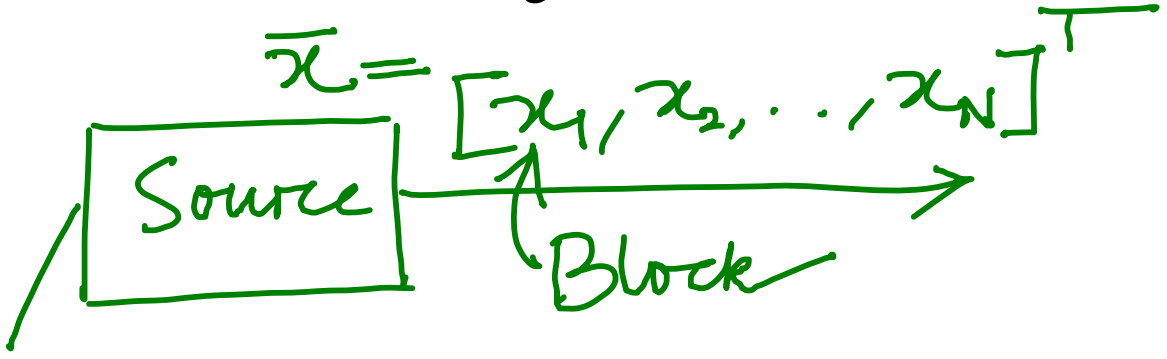
Encode the block of n symbols.

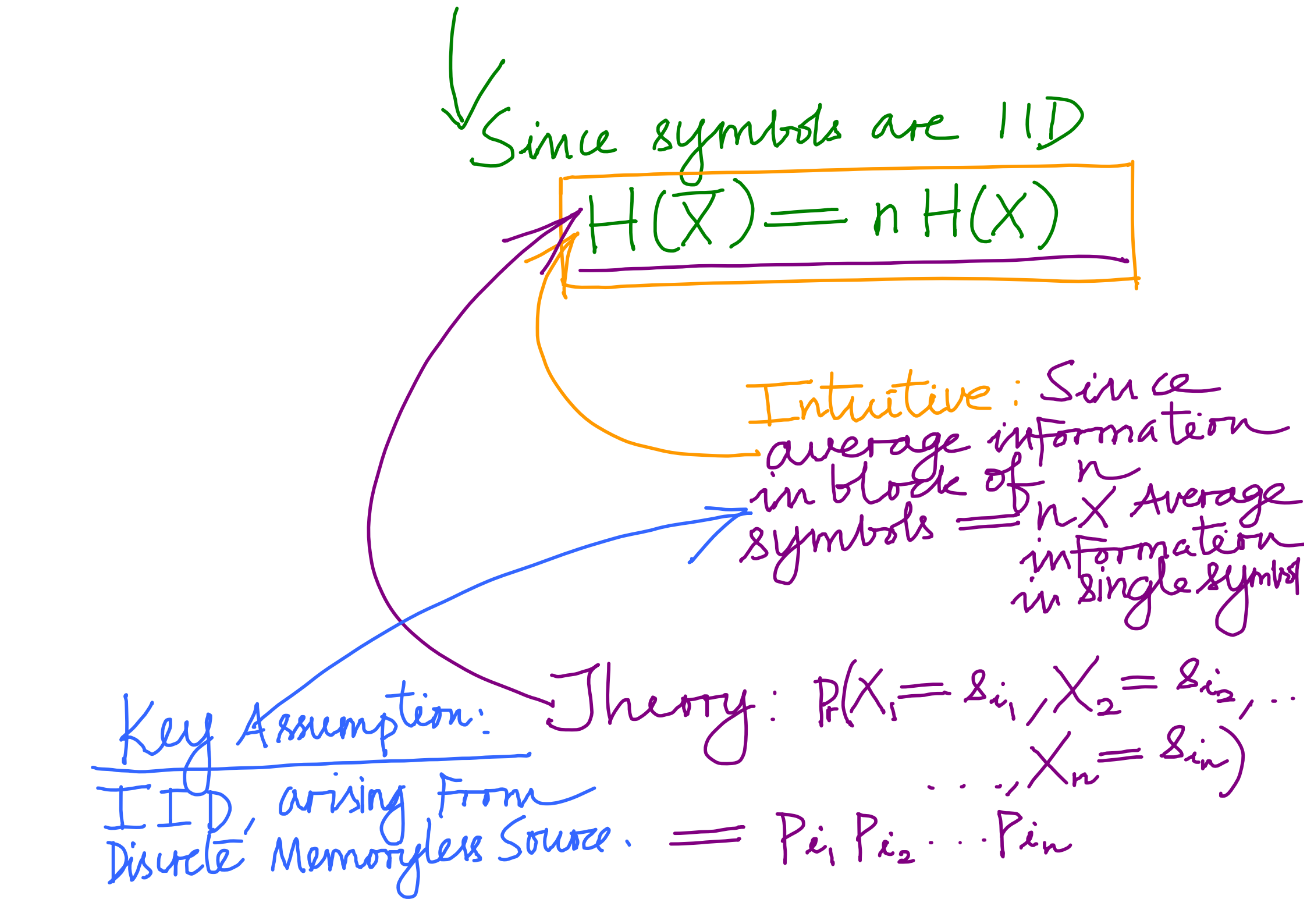
Discrete Memoryless Source

$\Rightarrow$ Symbols are I.I.D.

Let average codelength be L_n ← when encoding blocks of size n.

$H(X)$ = entropy of block source.




Since symbols are IID
$H(\bar{X}) = n\,H(X)$
Intuitive: Since average information in block of n symbols = n X Average information in single symbol
Theory: $P_r(X_1 = s_{i_1}, X_2 = s_{i_2}, \ldots, X_n = s_{i_n}) = P_{i_1} P_{i_2} \ldots P_{i_n}$
Key Assumption:
IID, arising From Discrete Memoryless Source.

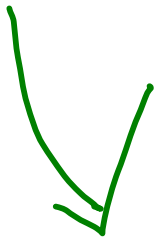
$$H(X) \leq L_n \leq H(X) + 1$$

$$nH(X) \leq L_n \leq nH(X) + 1$$

$$H(X) \leq \frac{L_n}{n} \leq H(X) + \frac{1}{n}$$

within $\frac{1}{n}$ of Entropy.

$\frac{L_n}{n}$ = $\frac{\#\text{bits per } n \text{ symbols}}{n}$ = Average number of bits per symbol.



By increasing n, one can make average code length arbitrarily close to Entropy.

Coding over larger & larger blocks.

i.e. within $\frac{1}{n}$ bits

and $\lim_{n \to \infty} \frac{1}{n} \to 0$

$\Rightarrow$ Gap between L_n and $H(X)$ approaches 0.